

# A NOTE ON THE NONEXISTENCE OF QUASI-HARMONIC SPHERES

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**ABSTRACT.** In this paper we study the properties of quasi-harmonic spheres from  $\mathbb{R}^m, m > 2$ . We show that if the universal covering  $\tilde{N}$  of  $N$  admits a nonnegative strictly convex function  $\rho$  with the exponential growth condition  $\rho(y) \leq C \exp(\frac{1}{4}\tilde{d}(y)^{2/m})$  where  $\tilde{d}(y)$  is the distance function on  $\tilde{N}$ , then  $N$  does not admit a quasi-harmonic sphere, which generalize Li-Zhu's result<sup>[8]</sup>. We also show that if  $u$  is a quasi-harmonic sphere, then the property that  $u$  is of finite energy ( $\int_{\mathbb{R}^m} e(u)e^{-|x|^2/4} dx < \infty$ ) is equivalent to the property that  $u$  satisfies the large energy condition ( $\lim_{R \rightarrow \infty} R^m e^{-R^2/4} \int_{B_R(0)} e(u)e^{-|x|^2/4} dx = 0$ ).

## 1. INTRODUCTION

Let  $M^m, N^n$  be two compact Riemannian manifolds of dimension  $m$  and  $n$  respectively. Let  $u \in W^{1,2}(M, N)$ , the energy of  $u$  is defined by

$$E(u) = \frac{1}{2} \int_M |du|^2 \, d \text{Vol}_M.$$

The critical points of the energy functional are called harmonic maps. Eells and Sampson<sup>[4]</sup> introduce the heat flow and prove that, the heat flow has a global solution which subconverges strongly to a harmonic map at infinity if the sectional curvature of the target manifold is non-positive. This result was generalized by Ding and Lin<sup>[3]</sup> to the case that the universal covering of  $N$  admits a nonnegative strictly convex function with quadratic growth.

However, in general, the heat flow may produce singularities at a finite time (e.g.<sup>[1;2]</sup>). Struwe divided singularities of the heat flow into two different types. One of this type is associated to quasi-harmonic spheres (c.f.<sup>[9]</sup>).

**Definition 1.1.** A quasi-harmonic sphere is a harmonic map from  $(\mathbb{R}^m, \exp(-x^2/2(m-2))g_0)$  to a Riemannian manifold, where  $g_0$  is the Euclidean metric in  $\mathbb{R}^m$  ( $m > 2$ ), i.e.,

$$(1.1) \quad \tau(u) = \frac{1}{2} x \cdot du,$$

with finite energy

$$(1.2) \quad \int_{\mathbb{R}^m} e(u)e^{-|x|^2/4} dx < \infty,$$

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where

$$e(u) = \frac{1}{2} |du|^2.$$

Based on the work of Lin and Wang<sup>[9]</sup>, we know that Liouville theorems for harmonic spheres (harmonic maps from spheres) and quasi-harmonic spheres imply the global existence of the heat flows. Li and Wang<sup>[6]</sup> proved that there are no non-constant quasi-harmonic spheres with images in a regular ball. Li and Zhu<sup>[8]</sup> proved that, if the heat flow has a global solution and there is no harmonic map from  $S^l$  to  $N$  for  $2 \leq l \leq m-1$ , then this flow subconverges in  $C^2$  norm to a smooth harmonic map at infinity. Moreover, in the same paper, they also proved that the heat flow exists globally provided that the universal covering  $\tilde{N}$  of  $N$  admits a strictly convex positive function  $\rho$  with polynomial growth, i.e.,

$$\tilde{\nabla}^2 \rho > 0, \quad 0 < \rho(y) < C(1 + \tilde{d}(y, y_0))^P, \quad \forall y \in \tilde{N},$$

for some  $y_0 \in \tilde{N}$  and some positive constants  $C, P$ . Here  $\tilde{d}$  is the distance function on  $\tilde{N}$ . Li and Yang<sup>[7]</sup> generalized these results to the case of “quasi-harmonic sphere with large energy condition” under the same assumption on  $\rho$ . The large energy condition is defined by

$$(1.3) \quad \lim_{R \rightarrow \infty} R^m e^{-R^2/4} \int_{B_R(0)} e(u) e^{-|x|^2/4} dx = 0.$$

Our first main result is as follows.

**Theorem 1.1.** *Suppose  $u$  satisfies (1.1), then the following three conditions are equivalent to each other.*

- (1) *The large energy condition holds, i.e., (1.3) holds.*
- (2)

$$\int_{\mathbb{R}^m} |u_r|^2 |x|^{4-m} dx < \infty.$$

- (3) *The total energy is finite, i.e., (1.2) holds.*

*Remark 1.1.* Li and Zhu<sup>[8]</sup> stated the following estimate for quasi-harmonic sphere,

$$(1.4) \quad \int_{B_R(0)} |du|^2 dx \leq CR^{m-2}, \quad \forall R > 0,$$

where  $C$  is a constant independent of  $R$ . As a consequence, this condition (1.4)<sup>1</sup> is equivalent to (1.2) and is also equivalent to the following condition

$$\int_{\mathbb{R}^m} |du|^2 |x|^{2-m-\delta} dx < \infty$$

for some or every  $\delta > 0$ . In fact, one can get more, see Corollary 2.5.

Our second main result is that, Li-Zhu’s result holds, if the universal covering  $\tilde{N}$  of  $N$  admits a nonnegative strictly convex function  $\rho$  with the following exponential growth condition: for some constant  $C$ ,

$$(1.5) \quad \rho(y) \leq C \exp\left(\frac{1}{4} \tilde{d}(y)^{2/m}\right), \quad \forall y \in \tilde{N}.$$

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<sup>1</sup>We thank ZHU Xiangrong for pointing out this equivalent condition.

Here  $\tilde{d}(y) = \tilde{d}(y, y_0)$  is the distance function on  $\tilde{N}$  from some fixed point  $y_0 \in \tilde{N}$ . It is easy to check that this assumption is weaker than the one in<sup>[8]</sup>.

**Theorem 1.2.** *Suppose  $m \geq 3$  and there is a nonnegative strictly convex function  $\rho$  on the universal covering of the target manifold  $N$  such that (1.5) holds. Then there is no non-constant quasi-harmonic sphere  $u$  from  $\mathbb{R}^m$  to  $N$ .*

## 2. PROOF OF THEOREM 1.1

In this section, we derive some estimates and prove Theorem 1.1. Introduce

$$H(r) := \int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta, \quad \forall r > 0.$$

We begin with the following Lemma.

**Lemma 2.1.** *Suppose  $u$  satisfies (1.1). Then*

(1) *either*

$$(2.1) \quad -R^{-2}(m-2) \int_{B_{\sqrt{2(m-2)}}} r^{2-m} |u_r|^2 dx \leq H(R) \leq 0, \quad \forall R > 0,$$

(2) *or there exists  $R_0 \geq \sqrt{2(m-2)}$  such that*

$$(2.2) \quad H(R) \geq R^{2-2m} e^{R^2/2} R_0^{2m-2} e^{-R_0^2/2} H(R_0) > 0, \quad \forall R > R_0.$$

Here  $S^{m-1}$  stands for the unit sphere in  $\mathbb{R}^m$  centering at 0 and  $B_R = B_R(0)$ .

*Proof.* A direct computation gives (c.f. Lemma 3.3 in<sup>[8]</sup>)

$$(2.3) \quad \frac{d}{dr} \int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta - \int_{S^{m-1}} \left( \frac{2}{r} e(u) + \left( \frac{r}{2} - \frac{m}{r} \right) |u_r|^2 \right) d\theta = 0, \quad \forall r > 0.$$

According to this identity, we get

$$\frac{d}{dr} \int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta + \frac{2}{r} \int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta = \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 d\theta.$$

From this formula, we know

$$(2.4) \quad \frac{d}{dr} (r^2 H(r)) = r^2 \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 d\theta.$$

Thus,  $r^2 H(r)$  is increase from  $\sqrt{2(m-2)}$  to infinity, and is decrease from 0 to  $\sqrt{2(m-2)}$ . Setting  $C_0 := \sqrt{2(m-2)}$ , we get

$$r^2 H(r) \geq C_0^2 H(C_0), \quad \forall r > 0.$$

Again according to (2.3) to obtain

$$\begin{aligned} & \frac{d}{dr} \int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta + \left( \frac{2(m-1)}{r} - r \right) \int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta \\ &= \left( r - \frac{2(m-2)}{r} \right) \int_{S^{m-1}} \left( e(u) - \frac{1}{2} |u_r|^2 \right) d\theta, \end{aligned}$$

which implies

$$(2.5) \quad \frac{d}{dr} \left( r^{2m-2} e^{-r^2/2} H(r) \right) = r^{2m-2} e^{-r^2/2} \left( r - \frac{2m-4}{r} \right) \int_{S^{m-1}} \left( e(u) - \frac{1}{2} |u_r|^2 \right) d\theta.$$

Hence,  $r^{2m-2} e^{-r^2/2} H(r)$  is increase from  $\sqrt{2(m-2)}$  to infinity, and is decrease from 0 to  $\sqrt{2(m-2)}$ . It is obvious that

$$r^{2m-2} e^{-r^2/2} \int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta \rightarrow 0, \quad \text{as } r \rightarrow 0.$$

Moreover,

$$\frac{d}{dr} (r^2 H(r)) \geq -(m-2)r \int_{S^{m-1}} |u_r|^2 d\theta,$$

which yields

$$R^2 H(R) \geq -(m-2) \int_{B_R} r^{2-m} |u_r|^2 dx, \quad \forall R > 0.$$

Here we have used the fact

$$\lim_{r \rightarrow 0} r^2 H(r) = 0.$$

Therefore,

$$r^2 H(r) \geq C_0^2 H(C_0) \geq -(m-2) \int_{B_{C_0}} r^{2-m} |u_r|^2 dx, \quad \forall r > 0.$$

Now we can finish the proof of this Lemma. If we do not have (2.1), then there exists  $R_0 \geq \sqrt{2(m-2)}$ , such that

$$\int_{\{R_0\} \times S^{m-1}} (|u_r|^2 - e(u)) d\theta > 0,$$

then for every  $r > R_0$ ,

$$r^{2m-2} e^{-r^2/2} H(r) \geq R_0^{2m-2} e^{-R_0^2/2} H(R_0) > 0,$$

which means that (2.2) holds. □

*Remark 2.1.* Suppose  $u$  satisfies (1.1), then

$$(2.6) \quad -R^2 H(R) \leq (m-2) \int_{B_{\sqrt{2(m-2)}}} r^{2-m} |u_r|^2 dx,$$

$$(2.7) \quad -R^{2m-2} e^{-R^2/2} H(R) \leq (m-2) \int_{B_{\sqrt{2(m-2)}}} r^{m-2} e^{-r^2/2} \frac{|u_\theta|^2}{r^2} dx,$$

$$(2.8) \quad -R^m e^{-R^2/4} H(R) \leq (m-2) \int_{B_{\sqrt{2(m-2)}}} e^{-r^2/4} e(u) dx,$$

holds for all  $R > 0$ .

*Proof.* The proof of (2.6) and (2.7) can be found in the proof of Lemma 2.1. The proof of (2.8) can be proved similarly since (2.3) implies the following formula

$$\frac{d}{dr} \left( r^m e^{-r^2/4} H(r) \right) = \left( \frac{r}{2} - \frac{m-2}{r} \right) r^m e^{-r^2/4} \int_{S^{m-1}} e(u) d\theta, \quad \forall r \in (0, \infty).$$

□

**Lemma 2.2.** *Suppose  $u$  satisfies (1.1) and*

$$\liminf_{R \rightarrow \infty} R^{2m-2} e^{-R^2/2} \int_{\{R\} \times S^{m-1}} (|u_r|^2 - e(u)) d\theta > 0,$$

*then*

$$\liminf_{R \rightarrow \infty} R^m e^{-R^2/4} \int_{B_R} (|u_r|^2 - e(u)) e^{-r^2/4} dx > 0.$$

*Proof.* A direct computation. □

Next, we prove the following energy estimate.

**Proposition 2.3.** *Suppose  $u$  satisfies (1.1), then there is a constant  $C_1$  depending only on  $m$  such that for every  $0 \leq \delta \leq 2$ , we have*

$$\int_{B_R} r^{4-m-\delta} |u_r|^2 dx \leq C_1 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 dx + 4R^2 H(R)^+, \quad \forall R > 0.$$

Here  $f^+ = \max\{f, 0\}$ .

*Proof.* We only consider the case  $R > 2\sqrt{m-2}$  and start with the formula (2.4), i.e.,

$$\frac{d}{dr} (r^2 H(r)) = r^2 \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 d\theta.$$

For every  $0 < \rho < R$ , we have

$$\begin{aligned} R^2 H(R) - \rho^2 H(\rho) &= \int_{\rho}^R r^2 \left( \frac{r}{2} - \frac{m-2}{r} \right) \int_{S^{m-1}} |u_r|^2 d\theta dr \\ &= \int_{B_R \setminus B_{\rho}} \left( \frac{r}{2} - \frac{m-2}{r} \right) r^{3-m} |u_r|^2 dx. \end{aligned}$$

For  $\sqrt{4(m-2)} \leq \rho < R$ , we have

$$\int_{B_R \setminus B_{\rho}} r^{4-m} |u_r|^2 dx \leq 4R^2 H(R)^+ - 4\rho^2 H(\rho),$$

which implies

$$\begin{aligned} \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{4-m} |u_r|^2 dx &\leq 4R^2 H(R)^+ - 4 \left( 2\sqrt{m-2} \right)^2 H \left( 2\sqrt{m-2} \right) \\ &\leq 4R^2 H(R)^+ + 4(m-2) \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 dx. \end{aligned}$$

Here we have used (2.6). In particular, we get the desired estimate for  $\delta = 0$ . In general  $0 \leq \delta \leq 2$ ,

$$\begin{aligned}
\int_{B_R} r^{4-m-\delta} |u_r|^2 \, dx &= \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{4-m-\delta} |u_r|^2 \, dx + \int_{B_{2\sqrt{m-2}}} r^{4-m-\delta} |u_r|^2 \, dx \\
&\leq \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{4-m} |u_r|^2 \, dx + \left(2\sqrt{m-2}\right)^{2-\delta} \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx \\
&\leq 8(m-2) \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+.
\end{aligned}$$

□

As a consequence,

**Corollary 2.4.** *Suppose  $u$  satisfies (1.1). Then there is a constant  $C_2$  such that for every  $0 < \delta < 1$ ,*

$$\delta R^{-\delta} \int_{B_R} r^{2-m+\delta} e(u) \, dx \leq C_2 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+, \quad \forall R > 0.$$

In particular,

$$(2.9) \quad R^{2-m} \int_{B_R} e(u) \, dx \leq C_2 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+, \quad \forall R > 0.$$

*Proof.* Since

$$\begin{aligned}
\int_{B_R} r^{2-m+\delta} e(u) \, dx &= - \int_0^R r^{1+\delta} H(r) \, dr + \int_{B_R} r^{2-m+\delta} |u_r|^2 \, dx \\
&\leq \sup_{0 < r < R} (-r^2 H(r)) \times \int_0^R r^{\delta-1} \, dr + R^\delta \int_{B_R} r^{2-m} |u_r|^2 \, dx \\
&= \sup_{0 < r < R} (-r^2 H(r)) \times \frac{R^\delta}{\delta} + R^\delta \int_{B_R} r^{2-m} |u_r|^2 \, dx.
\end{aligned}$$

Now applying Lemma 2.1 and Proposition 2.3, there exists a constant  $C_2$  depending only on  $m$  such that

$$\delta R^{-\delta} \int_{B_R} r^{2-m+\delta} e(u) \, dx \leq C_2 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx + 4R^2 H(R)^+.$$

□

Also, we can prove the following

**Corollary 2.5.** *Suppose  $u$  satisfies (1.1), then there is a constant  $C_3$  depending only on  $m$  such that for every  $0 < \delta < 1$ ,*

$$\delta \int_{B_R} r^{2-m-\delta} e(u) \, dx \leq C_3 \int_{B_{2\sqrt{m-2}}} r^{1-m} e(u) \, dx + 4R^2 H(R)^+, \quad \forall R > 0.$$

*Proof.* Similar to the proof of Corollary 2.4, for  $0 < \delta < 1$  and  $R > 2\sqrt{m-2}$ ,

$$\begin{aligned} \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m-\delta} e(u) \, dx &= - \int_{2\sqrt{m-2}}^R r^{1-\delta} H(r) \, dr + \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m-\delta} |u_r|^2 \, dx \\ &\leq \sup_{2\sqrt{m-2} < r < R} (-r^2 H(r)) \times \int_{2\sqrt{m-2}}^R r^{\delta-1} \, dr + \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, dx \\ &\leq \sup_{2\sqrt{m-2} < r < R} (-r^2 H(r)) \times \frac{2\sqrt{m-2}}{\delta} + \int_{B_R} r^{2-m} |u_r|^2 \, dx. \end{aligned}$$

Then Lemma 2.1 and Proposition 2.3 gives the desired estimate.  $\square$

Now we prove Theorem 1.1.

*Proof of Theorem 1.1.* Suppose the large energy condition holds, i.e., the claim (1) is true. Then according to Lemma 2.1 and Lemma 2.2 (or c.f. [7]), we know that  $H(r) \leq 0$  for every  $r > 0$ . Now the claim (2) follows from Proposition 2.3.

From the claim (2) to the claim (3), we need only to prove that

$$\int_{\mathbb{R}^m} r^{2-m-\delta} |du|^2 \, dx < \infty.$$

holds for some  $\delta > 0$ . According to Corollary 2.5, we need only to claim that  $\liminf_{R \rightarrow \infty} R^2 H(R)^+ \leq 0$ . This is true because

$$\liminf_{R \rightarrow \infty} R^2 H(R)^+ \leq \liminf_{R \rightarrow \infty} \int_{\{R\} \times S^{m-1}} |u_r|^2 \, d\theta$$

and the claim (2) implies the righthand is zero.

From the claim (3) to the claim (1) is obvious.  $\square$

### 3. PROOF OF THEOREM 1.2

The following Lemma is proved in [8]. Here we provide another proof which is simpler for  $m > 2$ .

**Lemma 3.1.** *Suppose  $f$  is a non-constant nonnegative smooth function satisfying*

$$\Delta f \geq \frac{1}{2} r f_r,$$

*then there exists a constant  $C > 0$  such that for  $r$  large enough,*

$$\int_{S^{m-1}} f(r, \theta) \, d\theta > C r^{-m} e^{r^2/4}.$$

*Proof.* Let

$$v(r) = \int_{S^{m-1}} f(r, \theta) \, d\theta,$$

then a direct computation yields

$$\frac{d}{dr} \left( r^{m-1} e^{-r^2/4} \frac{d}{dr} v \right) \geq 0.$$

Since  $\frac{dv}{dr} = O\left(\frac{1}{r}\right)$  as  $r \rightarrow 0$ , we obtain

$$\lim_{r \rightarrow 0} r^{m-1} e^{-r^2/4} \frac{d}{dr} v = 0,$$

since  $m > 2$ . In particular,

$$r^{m-1} e^{-r^2/4} \frac{d}{dr} v \geq 0.$$

Since  $f$  is not a constant, there exists  $a > 0$  such that  $\frac{dv}{dr}|_a > 0$ . The rest of the proof is simple (c.f. [8]).  $\square$

Let  $d(x) = \text{dist}(u(x), u(0))$ , then we have the following

**Lemma 3.2** (Refine energy estimate). *Suppose  $u$  is a quasi-harmonic sphere, then there is a constant  $C_m$  depending only on  $m$  such that for all  $R > 0$ ,*

$$\begin{aligned} \int_{B_R} d^2 dx &\leq C_m R^m \int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2 dx, \\ \int_{B_R} |\nabla d|^2 dx &\leq C_m R^{m-2} \int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2 dx. \end{aligned}$$

*Remark 3.1.* (1) Denoted  $E_R(u)$  by the energy of  $u$  on  $B_R$ , i.e.,

$$E_R(u) = \frac{1}{2} \int_{B_R} |du|^2 e^{-x^2/4} dx.$$

Then apply Corollary 2.5 to this Lemma to obtain the following estimate

$$\begin{aligned} \int_{B_R} d^2 dx &\leq C_m R^m E_R(u), \\ \int_{B_R} |\nabla d|^2 dx &\leq C_m R^{m-2} E_R(u). \end{aligned}$$

(2) Li and Zhu (c.f. Lemma 3.2 in [8]) obtained a similar result with constant  $C_{m,u}$  depending only on  $m$  and the total energy of  $u$  such that

$$\begin{aligned} \int_{B_R} d^2 dx &\leq C_{m,u} R^m, \\ \int_{B_R} |\nabla d|^2 dx &\leq C_{m,u} R^{m-2}. \end{aligned}$$

*Proof of Lemma 3.2.* It is clear that

$$d(r, \theta) \leq \int_0^r |u_s(s, \theta)| ds, \quad |\nabla d| \leq |du|.$$

Since the total energy of  $u$  is finite, by Lemma 2.2, we have

$$\int_{S^{m-1}} (|u_r|^2 - e(u)) d\theta \leq 0, \quad r > 0.$$



Applying (2.9), we obtain

$$\int_{B_R} |\nabla d|^2 \leq 2C_2 R^{m-2} \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 dx, \quad R > 0.$$

Next, we show

$$\int_{S^{m-1}} \left( \int_0^r |u_s(s, \theta)| ds \right)^2 d\theta \leq C_m \int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2 dx, \quad \forall r > 0.$$

Then the first part of this Lemma follows from this inequality. Without loss of generality, assume  $r > 1$ . Applying Proposition 2.3 and taking  $\delta = 1/2$ , we get

$$\int_{B_R} r^{7/2-m} |u_r|^2 dx \leq C_1 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 dx, \quad R > 0.$$

Using Minkowski's inequality, we get

$$\begin{aligned} \left( \int_{S^{m-1}} \left( \int_0^r |u_s(s, \theta)| ds \right)^2 d\theta \right)^{1/2} &\leq \int_0^r \left( \int_{S^{m-1}} |u_s(s, \theta)|^2 d\theta \right)^{1/2} ds \\ &\leq \int_0^1 \left( \int_{S^{m-1}} |u_s(s, \theta)|^2 d\theta \right)^{1/2} ds + \int_1^r \left( \int_{S^{m-1}} |u_s(s, \theta)|^2 d\theta \right)^{1/2} ds \\ &\leq \left( \int_0^1 \int_{S^{m-1}} |u_s(s, \theta)|^2 d\theta ds \right)^{1/2} \\ &\quad + \left( \int_1^r s^{5/2} \int_{S^{m-1}} |u_s|^2 d\theta ds \right)^{1/2} \left( \int_1^r s^{-5/2} ds \right)^{1/2} \\ &\leq C_m \left( \int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2 dx \right)^{1/2}. \end{aligned}$$

□

**Lemma 3.3.** *Suppose  $u$  is a quasi-harmonic sphere, then there is a constant  $C_m$  depending only on  $m$  such that*

$$\int_{B_r} \exp(C_m^{-1} E_r(u)^{-1/2} r^{2-m} d) dx \leq C_m, \quad \forall r > 1.$$

*Proof.* By the energy estimate Corollary 2.5, using an argument similar to the one used in the proof of Lemma 3.5 in [8], we can prove that the BMO subnorm  $[d]_{*, B_{2r}}$  of  $d$  over  $B_{2r}$  satisfies

$$(3.1) \quad [d]_{*, B_{2r}} := \sup_{x \in Q \subset B_{2r}} \int |d(y) - d_Q| dy \leq C_m \sqrt{E_{2r}(u)} (1+r)^{m-2},$$

where the supremum is taken over all cubes  $x \in Q \subset B_{2r}$ . The John-Nirenberg theorem (c.f. Lemma 1 in [5]) claims that there are two constants  $C_5, C_6$  depending only on  $m$  such that for all cubes  $Q \subset B_{2r}$ ,

$$\left| \{x \in Q : |d(x) - d_Q| > s\} \right| \leq C_5 \exp\left(-\frac{C_6 s}{[d]_{*, B_{2r}}}\right) |Q|,$$

which implies

$$\int_{B_r} \exp\left(\frac{C_6 |d - d_{B_r}|}{2[d]_{*, B_r}}\right) dx \leq C_5, \quad \forall r > 0.$$

Since we have the estimate (3.1), as a consequence, there is a constant  $C_7$  which depends only on  $m$  such that

$$\int_{B_r} \exp\left(C_7^{-1} E_r(u)^{-1/2} r^{2-m} |d - d_{B_r}|\right) dx \leq C_7, \quad \forall r > 1.$$

Finally, according to Lemma 3.2, we can find a constant  $C_8$  depending only  $m$  such that

$$d_{B_r} := \int_{B_r} d dx \leq C_8 E_r(u)^{1/2}.$$

Therefore, we get the desired estimate.  $\square$

*Remark 3.2.* Checking the proof of Lemma 3.5 in<sup>[8]</sup> step by step, and using the argument mentioned above, one can prove the following refine estimate,

$$\int_{B_r} \exp\left(C_m^{-1} \tilde{E}_{2\sqrt{m-2}}(u)^{-1/2} r^{2-m} d\right) dx \leq C_m, \quad \forall r > 1.$$

Here

$$\tilde{E}_R(u) = \int_{B_R} r^{1-m} |u_r|^2 dx.$$

In fact, checking the proof (c.f. page 455 in<sup>[8]</sup>), the constants come from either Lemma 3.2 or  $\tilde{E}_{3m}(u)$  which can be controlled by  $\tilde{E}_{2\sqrt{m-2}}(u)$  thanks to Corollary 2.5. Hence one can prove the required refine BMO estimate (3.1).

Now we give a poof of Theorem 1.2.

*Proof of Theorem 1.2.* Let  $\tilde{N}$  be the universal covering of  $N$ . Let  $\tilde{u} : \mathbb{R}^m \longrightarrow \tilde{N}$  be a lift of  $u$  with  $\tilde{u} = u \circ \pi$  where  $\pi : \tilde{N} \longrightarrow N$  is the covering map. It is easy to see that

$$\int_{\mathbb{R}^m} e(\tilde{u}) e^{-|x|^2/4} dx < \infty.$$

Set  $f = \rho \circ \tilde{u}$ , then

$$\Delta f - \frac{1}{2} r \partial_r f = \tilde{\nabla}^2 \rho(\tilde{u})(d\tilde{u}, d\tilde{u}) > 0.$$

Fixed  $p > 0$ . Notice that there is a constant  $C > 0$  such that

$$(3.2) \quad \int_{B_{2R}} f^p dx = \int_{B_{2R}} (\rho \circ \tilde{u})^p dx \leq C^p \int_{B_{2R}} e^{\frac{p}{4} \tilde{d}^{2/m}} dx, \quad R > 0.$$

Applying Young's inequality,

$$A + B \geq (PA)^{1/P} (QB)^{1/Q}, \quad A, B > 0, \quad P, Q \geq 1, \quad 1/P + 1/Q = 1,$$

we obtain that for  $\tilde{\delta} = p/(2m)$ ,

$$\begin{aligned} \tilde{\delta} r^{2-m} \tilde{d} + \left(\frac{p}{4} - \tilde{\delta}\right) r^2 &= \frac{p}{2m} r^{2-m} \tilde{d} + \left(\frac{p}{4} - \frac{p}{2m}\right) r^2 \\ &= \frac{p}{4} \left( \frac{2}{m} r^{2-m} \tilde{d} + \frac{m-2}{m} r^2 \right) \\ &\geq \frac{p}{4} \left( r^{2-m} \tilde{d} \right)^{2/m} \left( r^2 \right)^{(m-2)/m} \\ &= \frac{p}{4} \tilde{d}^{2/m}. \end{aligned}$$

Therefore, according to (3.2), for  $R > 0$ , we have

$$(3.3) \quad \int_{B_{2R}} f^p \, dx \leq C^p \int_{B_{2R}} e^{\tilde{\delta} R^{2-m} \tilde{d}(\tilde{u}, y_0)} e^{(p/4 - \tilde{\delta}) R^2} \, dx = C^p \int_{B_{2R}} e^{2^{m-2} \tilde{\delta} (2R)^{2-m} \tilde{d}(\tilde{u}, y_0)} e^{(p/4 - \tilde{\delta}) R^2} \, dx.$$

We can choose  $p > 0$  sufficiently small so that

$$2^{m-2} \tilde{\delta} = 2^{m-3} m^{-1} p \leq C_m^{-1} E^{-1/2},$$

which is equivalent to

$$E \leq \frac{m^2}{4^{m-3} C_m^2 p^2}.$$

According to Lemma 3.3 and (3.3), we can see that

$$\int_{B_{2R}} f^p \, dx \leq C^p e^{(p/4 - \tilde{\delta}) R^2} \int_{B_{2R}} \exp\left(C_m^{-1} E^{-1/2} (2R)^{2-m} \tilde{d}(\tilde{u}, y_0)\right) \, dx \leq C^p C_m (2R)^m e^{(p/4 - p/(2m)) R^2}$$

holds for  $R$  large enough.

If  $f$  is not a constant, applying Lemma 3.1 we obtain that for  $R$  large enough,

$$\int_{B_R} f \, dx \geq C_u R^{-2} e^{R^2/4}.$$

Here  $C_u > 0$  is a constant which is independent of  $R$ . Since  $f \geq 0$  satisfies

$$\operatorname{div}\left(e^{-|x|^2/4} \nabla f\right) \geq 0,$$

applying Moser's iteration (c.f. page 167 in<sup>[7]</sup>), for every  $p > 0$ , there is a constant  $C_p > 0$  depending only on  $p, m$  such that

$$\oint_{B_R} f \, dx \leq C_p R^{m/p} \left( \int_{B_{2R}} f^p \, dx \right)^{1/p}$$

holds for  $R$  large enough. Consequently, for  $R$  large enough

$$(3.4) \quad \int_{B_{2R}} f^p \, dx \geq C_p^{-p} C_u^p R^{-(m+2)p-m} e^{pR^2/4}.$$

Together with (3.3) and (3.4), we know that

$$0 < C_p^{-p} C_u^p \leq C^p C_m^2 m^{2m+(m+2)p} e^{-pR^2/(2m)} \rightarrow 0, \quad \text{as } R \rightarrow \infty.$$

This contradiction means that  $f$  is a constant. Moreover, since  $\rho$  is a strictly convex function, we get that  $d\tilde{u} = 0$ , i.e.,  $\tilde{u}$  is a constant. As a consequence,  $u$  is a constant.

□

## REFERENCES

- [1] K. C. Chang, W. Y. Ding, and R. G. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom. **36** (1992), no. 2, 507–515. MR 1180392
- [2] J. M. Coron and J. M. Ghidaglia, *Explosion en temps fini pour le flot des applications harmoniques*, C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), no. 12, 339–344. MR 992088
- [3] W. Y. Ding and F. H. Lin, *A generalization of Eells-Sampson's theorem*, J. Partial Differential Equations **5** (1992), no. 4, 13–22. MR 1192714
- [4] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. MR 0164306 (29 #1603)
- [5] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR 0131498
- [6] J. Li and M. Wang, *Liouville theorems for self-similar solutions of heat flows*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 1, 207–221. MR 2471137
- [7] J. Li and Y. Y. Yang, *Nonexistence of quasi-harmonic spheres with large energy*, Manuscripta Math. **138** (2012), no. 1-2, 161–169. MR 2898752
- [8] J. Li and X. R. Zhu, *Non existence of quasi-harmonic spheres*, Calc. Var. Partial Differential Equations **37** (2010), no. 3-4, 441–460. MR 2592981 (2011a:58029)
- [9] F. H. Lin and C. Y. Wang, *Harmonic and quasi-harmonic spheres*, Comm. Anal. Geom. **7** (1999), no. 2, 397–429. MR 1685578

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